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TYPICALLY REAL MEAN UNIVALENT FUNCTIONS OF LARGE GROWTH

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ABSTRACT

Typically real normalized κ-dimensionally mean univalent functions f on $|z|<1$ are considered for which

$$\limsup_{r\uparrow 1}\{(1-r)^2 \max_{0\le\theta\le 2\pi} |f(re^{i\theta})|\}>0.$$

Let $s=\log f(z)$ for z in the unit disc cut along $(-1,0]$. A theorem is proved concerning the area of the Riemann surface over the s-plane which distinguishes the two cases $-1\leqq\kappa<+\infty$ and $\kappa=+\infty$.

1. We consider those functions f regular in $|z|<1$ which have developments

$$z+a_2z^2+a_3z^3+\cdots \tag{1}$$

about $z=0$. Denote by $n(w)$ the sum of the multiplicities of the zeros of $f(z)-w$ in $|z|<1$ and by $p(t)$ the integral mean of $n(w)$ around $|w|=t$. We shall suppose that f is κ-dimensionally mean univalent, i.e. for some $\kappa, -1\leqq \kappa\leqq+\infty$, and all $R>0$ we have

$$\int_0^R (p(t)-1)t^{\kappa}dt\leqq 0. \tag{2}$$

The cases $\kappa=+\infty, +1, -1$ correspond to circumferentially (i.e. $p(t)\leqq 1$ for all $t>0$), areally, logarithmically mean univalent functions, respectively.

The function f is said to be of maximal growth if

$$\alpha=\limsup_{r\uparrow 1}\left\{(1-r)^2\max_{0\le\theta\le 2\pi}|f(re^{i\theta})|\right\}>0.$$

Then, provided $\kappa>-1$, it is known that there is ϕ such that $(1-r)^2|f(re^{i\phi})|\to\alpha\ (r\uparrow 1)$ ([1], Theorem 2). We shall assume throughout that $\phi=0$.

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Let Σ be the strip $|\tau| < \pi$ in the $s = \sigma + i\tau$ plane and let D be the Riemann surface over the s-plane which is the image of $|z| < 1$ cut along $(-1, 0]$ under $s = \log f(z) = \sigma(z) + i\tau(z)$, the principal branch of the logarithm being taken. (Note: (1), (2) imply that f is non-zero on $0 < |z| < 1$.) Suppose $\hat{D}$ is the projection of D onto the s-plane.

THEOREM. *Let f be a typically real κ-dimensionally mean univalent function of maximal growth. Then, for fixed κ, the area of $\Sigma\setminus(\Sigma \cap \hat{D})$ is finite for each such function f if and only if $\kappa = +\infty$.*

2. In this section we assume $\kappa = +\infty$ and show the area of $\Sigma\setminus(\Sigma \cap \hat{D})$ is finite. Using [2], Lemma 4 we have that

$$\int_{R_0}^{\exp[\sigma(x)]} \{Rp(R)\}^{-1} dR + 2\log(1-x)$$

tends to a finite limit as $x \uparrow 1$, where R_0 is a fixed positive number. The hypotheses of [2], Lemma 4 are satisfied, as a discussion analogous to that on [2], p. 153 shows. The maximal growth hypothesis is

$$\sigma(x) + 2\log(1-x) \to \log\alpha \quad (x \uparrow 1),$$

and so

$$\int_{R_0}^{\exp[\sigma(x)]} \frac{1-p(R)}{Rp(R)} dR$$

tends to a finite limit as $x \uparrow 1$. Since f is of maximal growth, we know that $\liminf_{R\to\infty} p(R) = 1$ ([2], p. 182), and as $p(R) \leqq 1$ for all R, we deduce that

$$\int_{R_0}^{\infty} R^{-1}(1-p(R))dR$$

is convergent and this shows that the area of $\Sigma\setminus(\Sigma \cap \hat{D})$ over $\mathrm{Re}(s) > \log R_0$ is finite. By the $\frac{1}{4}$-Theorem for circumferentially mean univalent functions ([3], Theorem 5.3), we see that $\Sigma \cap \hat{D}$ coincides with Σ in $\mathrm{Re}(s) < -\log 4$. The result follows and shows incidentally that the area of $\hat{D}\setminus(\Sigma \cap \hat{D})$ is also finite.

3. To complete the proof of the theorem we take κ finite and construct a κ-dimensionally mean univalent function of maximal growth for which the area of $\Sigma\setminus(\Sigma \cap \hat{D})$ is infinite. If we write $\theta(\sigma) = 2\pi p(e^{\sigma})$ and $k = \kappa + 1$ in the valency condition (2), we obtain

(2)′
$$\int_{-\infty}^{\log R} (\theta(\sigma) - 2\pi)e^{k\sigma} d\sigma \leqq 0 \quad (\text{all } R > 0).$$

We will modify the strip Σ by occasional small contractions and expansions of the width. We start with one perturbation of $\partial\Sigma$ and perform the relevant computations. Let $\theta(\sigma)$ be the width of a strip-like domain D which we take to be symmetric about $\tau = 0$. We suppose $k > 0$ and that ε, h, H are positive. Define θ on $[0, 6\varepsilon]$ by

$$\theta(\sigma) = \begin{cases} 2\pi - 2\sigma h\varepsilon^{-1} & 0 \leqq \sigma \leqq \varepsilon, \\ 2\pi - 2h & \varepsilon \leqq \sigma \leqq 2\varepsilon, \\ 2\pi - 2(3\varepsilon - \sigma)h\varepsilon^{-1} & 2\varepsilon \leqq \sigma \leqq 3\varepsilon, \\ 2\pi + 2(\sigma - 3\varepsilon)H\varepsilon^{-1} & 3\varepsilon \leqq \sigma \leqq 4\varepsilon, \\ 2\pi + 2H & 4\varepsilon \leqq \sigma \leqq 5\varepsilon, \\ 2\pi + 2(6\varepsilon - \sigma)H\varepsilon^{-1} & 5\varepsilon \leqq \sigma \leqq 6\varepsilon. \end{cases}$$

We shall choose $H = H(\varepsilon, h, k)$ so that

(3) $$\int_0^{6\varepsilon} (\theta(\sigma) - 2\pi)e^{k\sigma}d\sigma = 0.$$

Now

$$\begin{aligned} \int_{3\varepsilon}^{6\varepsilon} (\theta(\sigma) - 2\pi)e^{k\sigma}d\sigma &= 2H\varepsilon^{-1}\int_{3\varepsilon}^{4\varepsilon} (\sigma - 3\varepsilon)e^{k\sigma}d\sigma + 2H\int_{4\varepsilon}^{5\varepsilon} e^{k\sigma}d\sigma \\ &\quad + 2H\varepsilon^{-1}\int_{5\varepsilon}^{6\varepsilon} (6\varepsilon - \sigma)e^{k\sigma}d\sigma \\ &= 2H\varepsilon^{-1}e^{3k\varepsilon}\int_0^{\varepsilon} \sigma e^{k\sigma}d\sigma + 2He^{3k\varepsilon}\int_{\varepsilon}^{2\varepsilon} e^{k\sigma}d\sigma \\ &\quad + 2H\varepsilon^{-1}e^{3k\varepsilon}\int_{2\varepsilon}^{3\varepsilon} (3\varepsilon - \sigma)e^{k\sigma}d\sigma \\ &= -h^{-1}He^{3k\varepsilon}\int_0^{3\varepsilon} (\theta(\sigma) - 2\pi)e^{k\sigma}d\sigma. \end{aligned}$$

Taking

(3)′ $$H = he^{-3k\varepsilon},$$

we satisfy (3).

Next we establish

(4) $$\int_0^{6\varepsilon} \left. \frac{1}{\theta(\sigma)} - \frac{1}{2\pi}\right\} d\sigma = 3\pi^{-2}hk\varepsilon^2(1 + o(1)),$$

as $\varepsilon, h \downarrow 0$ with $h = o(\varepsilon)$. The contribution to the l.h.s. of (4) from $[0, 3\varepsilon]$ is

$$\frac{1}{2\pi}\int_0^{\varepsilon}\left\{\left(1-\frac{\sigma h}{\varepsilon\pi}\right)^{-1}-1\right\}d\sigma+\frac{1}{2\pi}\int_{\varepsilon}^{2\varepsilon}\left(1-\frac{h}{\pi}\right)^{-1}-1\Big\}\,d\sigma$$

$$+\frac{1}{2\pi}\int_{2\varepsilon}^{3\varepsilon}\left\{\left(1-\frac{(3\varepsilon-\sigma)h}{\varepsilon\pi}\right)^{-1}-1\right\}d\sigma$$

$$=\frac{1}{\pi}\int_0^{\varepsilon}\left\{\left(1-\frac{\sigma h}{\varepsilon\pi}\right)^{-1}-1\right\}d\sigma+\frac{\varepsilon h}{2\pi(\pi-h)}.$$

Similarly we obtain the contribution from $[3\varepsilon, 6\varepsilon]$ and so the l.h.s. of (4) is equal to

$$\frac{\varepsilon}{2\pi}\left[\frac{h}{\pi-h}-\frac{H}{\pi+H}\right]+\frac{1}{\pi}\left[\frac{\varepsilon h}{2\pi}+\frac{\varepsilon h^2}{3\pi^2}+o(\varepsilon h^3)-\frac{\varepsilon H}{2\pi}+\frac{\varepsilon H^2}{3\pi^2}+o(\varepsilon H^3)\right]$$

and this leads to (4) after a calculation involving (3)′.

We construct an L-strip D with boundary inclination zero at $\sigma = +\infty$ which is symmetric about $\tau = 0$, i.e., a strip-like domain D with differentiable width $\theta(\sigma)$ for which

$$\frac{\theta(\sigma_2)-\theta(\sigma_1)}{\sigma_2-\sigma_1}\to 0$$

as $\sigma_1, \sigma_2 \to +\infty$ simultaneously. Take two sequences $\{\varepsilon_n\}_{n=1}^{\infty}$, $\{h_n\}_{n=1}^{\infty}$ of positive numbers which tend to zero in such a way that

$$h_n\varepsilon_n^{-1}\to 0 \qquad (n\to\infty). \tag{5}$$

Let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of numbers tending to $+\infty$ for which $\sigma_1 > 0$, $\sigma_{n+1} > \sigma_n + 6\varepsilon_n$ (all n). We define $\theta(\sigma)$ to be equal to 2π if

$$\sigma \notin \cup_{n=1}^{\infty}(\sigma_n, \sigma_n + 6\varepsilon_n).$$

For other values of σ we set

$$\theta(\sigma)=\begin{cases} 2\pi-2(\sigma-\sigma_n)h_n\varepsilon_n^{-1} & \sigma_n \leqq \sigma \leqq \sigma_n+\varepsilon_n,\\ 2\pi-2h_n & \sigma_n+\varepsilon_n \leqq \sigma \leqq \sigma_n+2\varepsilon_n,\\ 2\pi-2(\sigma_n+3\varepsilon_n-\sigma)h_n\varepsilon_n^{-1} & \sigma_n+2\varepsilon_n \leqq \sigma \leqq \sigma_n+3\varepsilon_n,\\ 2\pi+2(\sigma-\sigma_n-3\varepsilon_n)H_n\varepsilon_n^{-1} & \sigma_n+3\varepsilon_n \leqq \sigma \leqq \sigma_n+4\varepsilon_n,\\ 2\pi+2H_n & \sigma_n+4\varepsilon_n \leqq \sigma \leqq \sigma_n+5\varepsilon_n,\\ 2\pi+2(\sigma_n+6\varepsilon_n-\sigma)H_n\varepsilon_n^{-1} & \sigma_n+5\varepsilon_n \leqq \sigma \leqq \sigma_n+6\varepsilon_n, \end{cases}$$

where $H_n = h_n e^{-3k\varepsilon_n}$.

Because of (5), the domain D in the $s = \sigma + i\tau$ plane defined by $|\tau| < \frac{1}{2}\theta(\sigma)$ is an L-strip with boundary inclination zero at $\sigma = +\infty$.

Let g be the function regular in $|z| < 1$ and having normalization (1)[†] for which the principal branch of $\log g$ maps $|z| < 1$ cut along $(-1, 0]$ onto D so that $(0, 1)$ in $|z| < 1$ is mapped onto $\tau = 0$ in D. Property (3) and the definition of θ ensure that g is a typically real k-dimensionally mean univalent function in $|z| < 1$. Further, (4) shows that

(6) $$\int_0^{\sigma'} \frac{d\sigma}{\theta(\sigma)} - \int_0^{\sigma'} \frac{d\sigma}{2\pi}$$

converges as $\sigma' \to +\infty$, provided (5) holds and also

(7) $$\sum_{n=1}^{\infty} \varepsilon_n^2 h_n < \infty.$$

An application of Warschawski's first inequality ([4], p. 280) to the map of D onto Σ which takes the positive real axis onto itself yields

(8) $$\log \frac{x}{(1-x)^2} \leqq 2\pi \int_{\sigma_0}^{\log g(x)} \frac{d\sigma}{\theta(\sigma)} + \frac{2\pi}{12} \int_{\sigma_0}^{\log g(x)} \frac{(\theta'(\sigma))^2}{\theta(\sigma)} d\sigma + K,$$

where $0 < x < 1$, σ_0 is fixed and K is a constant depending on σ_0 and D. If we make

(9) $$\sum_{n=1}^{\infty} h_n^2 \varepsilon_n^{-1} < \infty,$$

then

$$\int^{\infty} \frac{(\theta'(\sigma))^2}{\theta(\sigma)} d\sigma < \infty,$$

and so (9), (6) when substituted into (8) produce

$$(1-x)^2 g(x) > \beta > 0$$

for some constant β independent of x. Thus, if we satisfy (5), (7), (9), g will be a κ-dimensionally mean univalent function in $|z| < 1$ of maximal growth along $\arg z = 0$.

The area of $\Sigma \setminus (\Sigma \cap D)$ will be infinite if

(10) $$\sum_{n=1}^{\infty} \varepsilon_n h_n = +\infty.$$

† To ensure that $g'(0) = 1$ a horizontal translation of D may be needed. We suppose the sequence $\{\sigma_n\}$ has been suitably modified.

The choice $h_n = \varepsilon_n^3$, $\varepsilon_n = n^{-1/4}$ satisfies (5), (7), (9), (10) and completes the construction for $\kappa > -1$. When $\kappa = -1$ any example with $\kappa > -1$ will suffice.

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